Mathematical Preliminaries

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Ito's Lemma

Suppose a random variable x follows the dynamics

$$dx = a(x,t)dt + b(x,t)dz$$

Here, dz follows a Wiener process.
And $a(x,t)$ and $b(x,t)$ are functions of x,t.

A Wiener process is a Markov stochastic process with these two properties.

1. $\delta z = \varepsilon\sqrt{\delta t}$ The variable ε is a $N(0,1)$ RV
2. The increments to Z are independent

Ito's Lemma (contd.)

The mean of δz = 0 and its Variance = δt

Ito's lemma shows that if G is a function of x and t, then

$$dG = (\frac{\partial g}{\partial x} a + \frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2)dt + \frac{\partial G}{\partial x} b dz$$

If x follows an Ito process , so will G

Markov Chains and Pushkin's Poetry

- The Russian mathematician Andrey Markov was trying to analyze whether to guess if in a document, the *k-th* letter would be a vowel or a constant.
- Markov analysed 20 thousand letters from Pushkin's poem Eugene Origin.
- Vowels occur 43% of the time; so always guessing consonant is right 57% of the time.
- But a vowel is followed by a consonant 87% of the time. A consonant is followed by a vowel 66% of the time.Therefore, knowledge of the preceding letter is very helpful! Reversal at work!
- But he found that knowledge of the preceding ***two*** letters did not confer any additional advantage.
- This leads to the central idea of a Markov Chain- while successive outcomes may not be independent, only the most recent outcome is helpful in predicting the next outcome

Simple Way to Think About Ito's Lemma

Consider G, a continuous function of x and δx is a small change in x *and* δG is the resulting small change in G.

$$\delta G \approx \frac{dG}{dx}\delta x$$

This ignores all terms of order δx^2 and higher.

Applying Taylor's Theorem,

$$\delta G = \frac{dG}{dx}\delta x + \frac{1}{2}\frac{d^2G}{dx^2}\delta x^2 + \frac{1}{3!}\frac{d^3G}{dx^3}\delta x^3 + \ldots\ldots..$$

Simple Way ... (contd.)

Applying Taylor series expansion to the

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\delta x^2 + \frac{\partial^2 G}{\partial x \partial y}\delta x \delta y + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}\delta y^2 + ...$$

As δx and δy tend to zero,this becomes

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy$$

Now we move to the case where variable ***x*** follows an Ito's process

$$\delta x = a\delta t + b\varepsilon\sqrt{\delta t}$$

$\delta x^2 = b^2\varepsilon^2\delta t$ + terms of order > 2 (in δt)

Simple Way ... (contd. 2)

Therefore........

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(b^2\varepsilon^2\delta t) + ...$$

$$dG = \frac{\partial G}{\partial x}dx + (\frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt$$

we use the fact that for a random variable ω *~N(0,1)*

$$E(\omega^2) - [E(\omega)]^2 = 1$$

ω

$E(\omega\ ) = 0,$ and $E(\omega^2) = 1,$ and

$\varepsilon^2\delta t$ can be thought of as a non-stochastic variable, equal to its expected value which is δt

So, we get the two forms of Ito's Lemma

$$dG = \frac{\partial G}{\partial x}dx + (\frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt$$

$$dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt + \frac{\partial G}{\partial x}bdz$$

A Tragic Tale!

- Wolfgang Doeblin was a German-French Jewish Mathematician. He studied Probability Theory under Frechet, and gained a strong reputation.
- In World War II, he joined the army and volunteered to fight at the front where his company surrendered
- Rather than be captured, he burned his notes and took his life - however most of his work had been sent to the Academie des Sciences de Paris and his notebook lay in a safe - secure but forgotten.
- This was opened in 2000 and it was realized that Doeblin had been 25 years ahead of everyone else in the theory of Markov Processes.
- The Ito-lemma is now called Ito-Doeblin to recognize his achievement.

Lognormal Distribution

Lognormal distribution :

A random variable X is said to follow a **Lognormal** distribution if its **Log**arithm is **Normal**ly distributed

pdf of X : $\frac{1}{(\sqrt{2\pi})\sigma x} e^{-\frac{\{(\log x - \mu)^2\}}{2\sigma^2}}$, x >0

μ = mean of log X, σ = std dev of log X

Mean of X = $e^{[\mu + \frac{1}{2}\sigma^2]}$ *(note the + symbol)*

Variance of X = $e^{(2\mu+\sigma^2)} . (e^{\sigma^2} - 1)$

Normal distribution

pdf : $\frac{1}{(\sqrt{2\pi})\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Applying Ito's Lemma to Log S

If $dS = S(\mu dt + \sigma dz)$ and $G = \ln S$

$$dG = (\mu - \frac{\sigma^2}{2})dt + \sigma dz$$

changes in *ln S* are normally distributed ; the drift is $(\mu - \frac{\sigma^2}{2})$ and the variance is σ^2

If you observe the change in Log (S) from 0 to *T* its mean is $(\mu - \frac{\sigma^2}{2})T$ & variance is $\sigma^2 T$.

This is used in Monte Carlo simulation of Geometric Brownian Motion

$$d \ln S = (\mu - \frac{\sigma^2}{2})dt + \sigma dz$$ *(note the - symbol)*

$$s(t + \Delta t) = s(t).e^{[(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}]}$$

Monte Carlo and Stanislaw Ulam

- Stanislaw Ulam was a Jewish mathematician from Poland who fled to the US on the eve of the World War.
- John von Neuman invited him to join a secret war project in New Mexico; he borrowed a book on New Mexico, looked at check out card and saw all the names of people who had left University of Wisconsin-Madison.
- Invented the monte Carlo technique while at Los Alamos for evaluating integrals.(Enrico Fermi had used this earlier).
- Worked on Hydrogen Bomb with Edward Teller.Improved on Teller's design; fission-fusion. Computer calculation...

Derivation of Black-Scholes-Merton Equation

We can apply Ito's Lemma to a derivative security V

$$dV = \left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) + \frac{\partial V}{\partial S}\sigma S dz$$

Now, if we are short 1 unit of V and long $\frac{\partial V}{\partial x}$ shares of S

then this portfolio is $\Pi = -V + \frac{\partial V}{\partial S} S$ ----(a)

$\delta\Pi = -\delta V + \frac{\partial V}{\partial S}.\delta S$ But recall $\delta S = S(\mu\, dt + \sigma\delta\, z)$

and therefore $\delta\Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)\delta t$ ----(b)

This does not involve the Wiener process term.Therefore it grows at the risk less rate: $\delta\Pi = r\Pi\delta t$

and substitute (a) and (b) into (c)

$$\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)\delta t = r\left(-V + \frac{\partial V}{\partial S}S\right)\delta t$$

or $$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) = r\left(V - \frac{\partial V}{\partial S}S\right)$$

or $$\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) = rV$$

This is Black-Scholes-Merton equation

Black - Scholes-Merton Formula

BSM Call $e^{-q\tau} S\Phi(d_1) - e^{-r\tau} K\Phi(d_2)$

BSM Put $e^{-r\tau} K\Phi(-d_2) - e^{-q\tau} S\Phi(-d_1)$

$$d_1 = \frac{\ln(S / X) + (r - \sigma^2 / 2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

Where S is the stock price, X is the strike price, r is the risk free rate and t is the time to maturity and the volatility of the underlying is represented by σ